Information-theoretic characterization of the Subregular Hierarchy

Huteng Dai[1] and Richard Futrell[2]

Rutgers University[1] and University of California, Irvine[2]

Goal

To **link two notions of complexity**:

- **Formal language theory** (FLT) (Hopcroft and Ullman, 1979; Heinz, 2018)
- **Statistical complexity theory** (Feldman and Crutchfield, 1998)

Why?

Statistical complexity theory quantifies the **memory requirements** for generating a sequence incrementally. Limitations of human working memory may explain why natural language occupies the formal complexity class that it does.

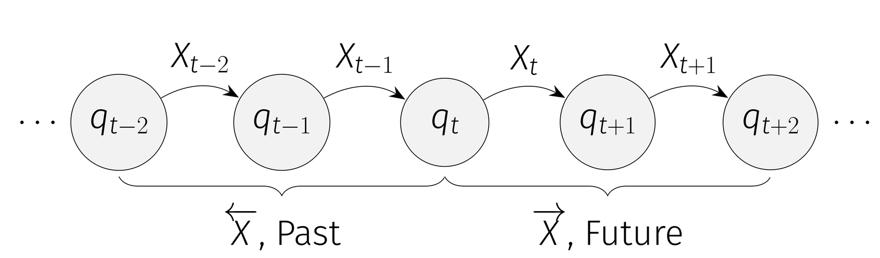
Probabilistic Deterministic Finite-State Automata (PDFAs)

A Probabilistic Deterministic Finite-State Automaton (PDFA) has:

- **Internal states** $\mathcal{Q}$, an **alphabet** Σ,
- an **emission distribution** $O : \mathcal{Q} \to \Sigma$ (a stochastic function), and
- a **transition function** $T : \mathcal{Q} \times \Sigma \to \mathcal{Q}$ (a deterministic function).

To construct a stochastic process from a PDFA: whenever the PDFA emits an end-of-word symbol, let it transition back into its initial state. The resulting infinite stream of symbols is a **stationary ergodic stochastic process**.

- Let X be any stochastic process generating sequences of symbols indexed as $\ldots X_{t-2}, X_{t-1}, X_t, X_{t+1}, \ldots$.



Statistical Complexity

The statistical complexity of X is the minimal amount of information about the past which must be stored in incremental memory in order to correctly generate the future:

$$S \equiv \min_{M : D_{KL}[\overrightarrow{X} | \overleftarrow{X} || \overrightarrow{X} | M] = 0} H[M],$$

$$\text{where } H[M] \equiv -\sum_x p_M(x) \log p_M(x).$$

Statistical complexity breaks into two terms (where $I[X : Y]$ is **mutual information**):

$$S = H[M] = I[M : \overrightarrow{X}] + H[M | \overrightarrow{X}]$$
$$= \underbrace{I[\overleftarrow{X} : \overrightarrow{X}]}_{E} + \underbrace{H[M | \overrightarrow{X}]}_{C},$$

Some Example PDFAs

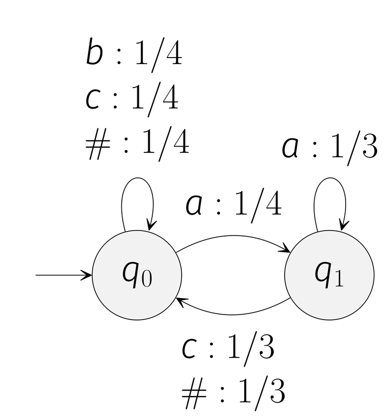


Figure 1: SL_2 PDFA of $\neg ab$, $\Sigma = \{a, b, c\}$

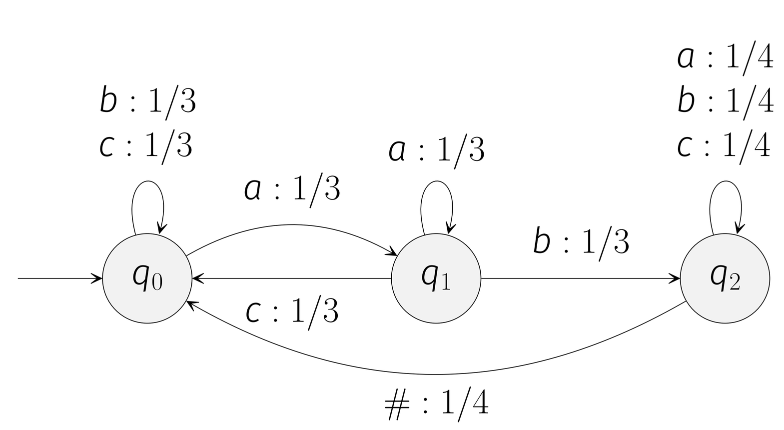


Figure 2: LT_2 PDFA of Some-ab, $\Sigma = \{a, b, c\}$

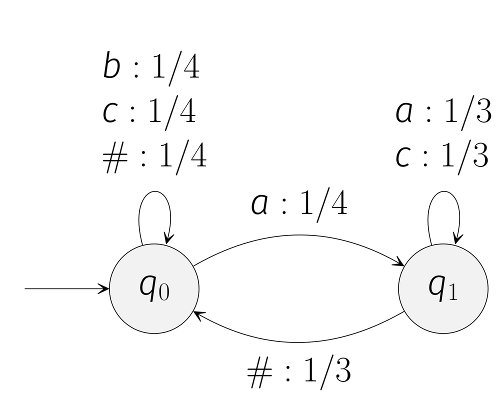


Figure 3: SP_2 PDFA of $\neg a \ldots b$, $\Sigma = \{a, b, c\}$

Calculation of Quantities for PDFAs

- The **statistical complexity** in a minimal PDFA is the entropy of the stationary distribution Q over states, given by the first left eigenvector of the state transition matrix, whose entries are given by:

$$p(q_{t+1} | q_t) = \sum_{x_t \in \Sigma} p_O(x_t | q_t) \delta_{q_{t+1} = T(q_t, x_t)}.$$

- For SL_k languages, **excess entropy**

$$E = I[X_{t-k+1}, \ldots, X_{t-1} : X_t, \ldots, X_{t+k-1}].$$

- For other languages, an SL_k approximation always gives a **lower bound on** E (Travers and Crutchfield, 2011). We compute lower bounds on E from SL_7 approximations.

Quantities

- **Excess entropy** E is the amount of information in the past which is actually used to predict the future.
- **Crypticity** C is the amount of information which must be stored, but which ends up being useless.
- **E/S ratio** is the proportion of bits stored in memory which end up being useful.

Main Result

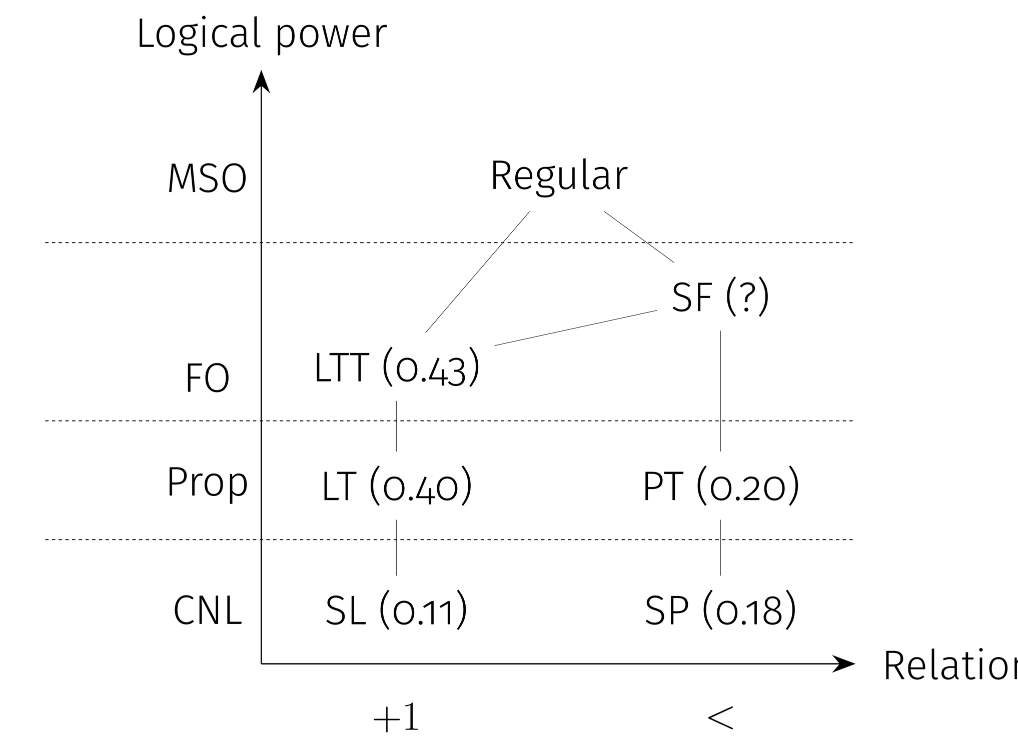
In a **series of example languages**, we find that increasing FLT complexity in the subregular hierarchy correlates with **E/S ratio**.

	SL_2	LT_2	LTT_2	SP_2	PT_2
S	0.97	1.53	1.94	0.99	1.53
E	0.09	$\geq$0.61	$\geq$0.83	$\geq$0.18	$\geq$0.30
C	0.75	$\leq$0.91	$\leq$1.10	$\leq$0.80	$\leq$1.22
E/S	0.11	$\geq$0.40	$\geq$0.43	$\geq$0.18	$\geq$0.20

Table 1: Information quantities for PDFAs

Conclusion

Based on our examples, we **conjecture** that crypticity correlates with logical power in the subregular hierarchy. In future work, we will formalize this idea.



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Acknowledgements

We thank Jim Crutchfield, Jeff Heinz, Adam Jardine, and anonymous reviewers for their comments and insights.

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